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Essential constants for spatially homogeneous Ricci-flat manifolds of dimension 4+1

T Christodoulakis[1], **S Hervik**[2] **and G O Papadopoulos**[1]

[1] Physics Department, University of Athens, Nuclear & Particle Physics Section, Panepistimioupolis, Ilisia GR 157-71, Athens, Hellas, Greece
[2] DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK

E-mail: tchris@cc.uoa.gr, S.Hervik@damtp.cam.ac.uk and gpapado@cc.uoa.gr

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Abstract
The present work considers (4+1)-dimensional spatially homogeneous vacuum cosmological models. Exact solutions—some already existing in the literature, and others believed to be new—are exhibited. Some of them are the most general for the corresponding Lie group with which each homogeneous slice is endowed, and some others are quite general. The characterization 'general' is given based on the counting of the essential constants, the line element of each model must contain; indeed, this is the basic contribution of the work. We give two different ways of calculating the number of essential constants for the simply transitive spatially homogeneous (4+1)-dimensional models. The first uses the initial value theorem; the second uses, through Peano's theorem, the so-called time-dependent automorphism inducing diffeomorphisms.

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1. Introduction

Since the dawn of *general relativity* people have been interested in finding exact solutions to Einstein's field equations[3]. However, due to the fairly complicated nature of the field equations, we usually impose symmetries in order to make the field equations more tractable. Some of the most successful schemes of symmetry reductions are the so-called Bianchi models in (3+1)-dimensional cosmology [2–4]. Here, in this paper we will consider their (4+1)-dimensional counterparts [5, 6].

The study of higher dimensional models has—especially since the advent of *string theory* [7, 8]—become increasingly popular in recent years. For example, exact solutions like

[3] See, e.g., [1] for the known exact solutions in 3+1 dimensions.

plane-wave spacetimes have been in focus for the last few years because they admit supersymmetry and they provide an exactly soluble string background. Plane-wave spacetimes are some of the solutions of the models considered here. More specifically, we will consider (4+1)-dimensional spatially homogeneous spacetimes which are solutions of the vacuum field equations. Equivalently, we will consider Ricci-flat spacetimes; i.e. spacetimes obeying

$$R_{ab} = 0 \tag{1.1}$$

which admit a group acting simply transitively on the spatial hypersurfaces. The question we address is 'how large is the set of Ricci-flat spacetimes within the set of models considered?' or equivalently 'how many parameters are necessary, in principle, to specify a solution to these equations provided that they are spatially homogeneous?'. The answer to this question turns out to be that 11 parameters need to be specified to give a solution for the most general classes.

Many interesting phenomena are related to this issue. For example, a by-product of our analysis is that we are able to determine which are the most general vacuum models within the class of spatially homogeneous models. In (3+1)-dimensional cosmology, the most general simply connected Bianchi vacuum models, namely type VIII, IX and the exceptional model $\mathrm{VI}^*_{-1/9}$ [4][4], are all chaotic in the initial singular regime [9–18]. In 4+1 cosmology one might wonder if the same is the case[5]. We will also give some exact solutions, thereby providing some examples of spacetimes of each class. In some cases, the entire family is known explicitly; in others only a few, or even none, are known. However, some of these special solutions have some interesting properties—such as self-similarity—which may be important in the late-time behaviour of more general solutions (see, e.g., [4]). As an explicit example of this, the plane-wave solutions—which will be discussed later—were shown in [20] to be the attractors within their class of models.

The paper is organized as follows. Next, we introduce the automorphism group and see how it is related to coordinate transformations, or the gauge freedom in general relativity (see, e.g., [21, 22]). Then, in section 3, we present our main results, namely the counting of the essential constants for the simply transitive, spatially homogeneous models of dimension 4+1. In section 4 we present some exact solutions before we conclude in section 5.

2. The role of the automorphism group

Let us first exhibit some basic assumptions that lie in the foundation of our work.

Spacetime is assumed to be the pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a five-dimensional, Hausdorff, connected, time-oriented and C^∞ manifold, and g is a (0, 2) tensor field, globally defined, C^∞, non-degenerate and Lorentzian (i.e. it has signature $(-, +, +, +, +)$). In the spirit of 4+1 analysis, we foliate the entire spacetime like $\mathcal{M} = \mathbb{R} \times \Sigma_t$, where the four-dimensional orientable submanifolds Σ_t (*surfaces of simultaneity*) are space-like surfaces of constant time. The assumption of spatial homogeneity corresponds to imposing the action of a symmetry group of transformations G upon the manifolds Σ_t. Usually, the group G is not only continuous, but also a Lie group—thus denoted by G_r, where r is the dimension of the space of its parameters. Avoiding the details on these issues—these matters can be easily found in a standard reference, see e.g. [23, 24]—we simply state that spatially homogeneous models with a simply transitive action of the group G_4 are described (apart from the topology of Σ_t which we will assume is simply connected) by an invariant basis of one-forms $\boldsymbol{\omega}^\alpha = \sigma^\alpha_i(x)\,\mathrm{d}\boldsymbol{x}^i$

[4] Strictly speaking, it depends how one counts. If we include the group parameter as an essential constant, then types VI_h and VII_h are equally general.
[5] This was recently addressed for some of the models in [19].

(their Lie derivative with respect to the generators of the Lie group G_4 $\xi^i_\alpha(x)$, are zero). There is also the case of spatially homogeneous models in which the group acts multiply transitively (the number of generators is more than 4 and there does not exist a proper invariant subgroup of dimension 4 acting transitively). The multiple transitive cases, which in dimension 4+1 are five in total [6], will not be considered here. More generally, a Lie group, G_r, is said to act transitively if the following requirements are satisfied:

(i) $r \geqslant$ dimension of Σ_t $(=4)$.
(ii) The rank of the matrix constructed by the generators (seen as vector fields) is everywhere equal to the dimension of Σ_t $(=4)$.

Geometrically, the above requirements imply that two different points in a given domain of Σ_t can be interchanged by a Lie group transformation. Simply transitive action, which is our concern in this paper, corresponds to the case $r =$ dimension of Σ_t $(=4)$. In this case we note that the most general line element, manifestly invariant under the action of the group, takes the form (in an appropriate coordinate system)[6]

$$\mathrm{d}s^2 = (N^\alpha(t)N_\alpha(t) - N^2(t))\,\mathrm{d}t^2 + 2N_\alpha(t)\sigma^\alpha_i(x)\,\mathrm{d}t\,\mathrm{d}x^i + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x)\,\mathrm{d}x^i\,\mathrm{d}x^j \qquad (2.1)$$

with

$$\sigma^\alpha_{i,j}(x) - \sigma^\alpha_{j,i}(x) = 2C^\alpha_{\mu\nu}\sigma^\mu_i(x)\sigma^\nu_j(x) \qquad (2.2)$$

where $\gamma_{\alpha\beta}(t)$ is the metric induced on the surfaces Σ_t (and thus constant on them), $N(t)$ is the lapse function, $N_\alpha(t)$ is the shift vector ($N^\alpha(t) = \gamma^{\alpha\beta}(t)N_\beta(t)$, $\gamma^{\alpha\beta}(t)$ being the inverse of $\gamma_{\alpha\beta}(t)$), and $C^\alpha_{\mu\nu}$ are the structure constants of the corresponding (closed) Lie algebra. In four dimensions, there are 30 closed, real, Lie algebras [25, 26].

At this point, a question arises: is there any particular class of general coordinate transformations (GCTs) which can serve to simplify the form of the line element and thus also Einstein's field equations (EFEs)? The answer is positive and a thorough investigation of this problem and its consequences is given in [22]; indeed, not only is there a class of GCTs which preserves the manifest spatial homogeneity of the line element (2.1), but it also forms a continuous (and virtually Lie) group. This group is closely related to the symmetries of the symmetry Lie group G_r; it is its automorphism group.

In the spirit of the 4+1 analysis we consider, apart from the time reparametrization, the following GCTs:

$$t \to \widetilde{t} = t \quad \Leftrightarrow \quad t = \widetilde{t} \qquad x^i \to \widetilde{x}^i = g^i(t, x^j) \quad \Leftrightarrow \quad x^i = f^i(t, \widetilde{x}^j). \qquad (2.3)$$

After insertion of (2.3) into (2.1), the wish to preserve the manifest homogeneity of the latter leads, in a first step, to the allocations

$$\frac{\partial f^i}{\partial t} = \sigma^i_\alpha(f)P^\alpha(t, \widetilde{x}) \qquad (2.4a)$$

$$\frac{\partial f^i}{\partial \widetilde{x}^j} = \sigma^i_\alpha(f)\Lambda^\alpha_\beta(t, \widetilde{x})\sigma^\beta_j(\widetilde{x}) \qquad (2.4b)$$

and, consequently,the definitions

$$\widetilde{N}(t) = N(t) \qquad (2.5a)$$

$$\widetilde{N}^\alpha(t) = S^\alpha_\beta(t)(N^\beta(t) + P^\beta(t, \widetilde{x})) \qquad (2.5b)$$

$$\widetilde{\gamma}_{\alpha\beta}(t) = \Lambda^\mu_\alpha(t, \widetilde{x})\Lambda^\nu_\beta(t, \widetilde{x})\gamma_{\mu\nu}(t) \qquad (2.5c)$$

[6] Greek indices label the invariant one-forms while the Latin indices run over the spatial coordinates (i.e. from 1–4).

where $N^{\alpha}(t,\widetilde{x}) = \gamma^{\alpha\beta}(t)N_{\beta}(t,\widetilde{x})$ with $\sigma^{i}_{\alpha}(x)$ being the inverses of $\sigma^{\alpha}_{i}(x)$—quantities which exist in the simply transitive cases. In order for the transformations (2.3) to have a well-defined non-trivial action, it is pertinent for the quantities Λ^{α}_{β} and P^{α} to be space independent. So,

$$\frac{\partial f^{i}}{\partial t} = \sigma^{i}_{\alpha}(f)P^{\alpha}(t) \tag{2.6a}$$

$$\frac{\partial f^{i}}{\partial \widetilde{x}^{j}} = \sigma^{i}_{\alpha}(f)\Lambda^{\alpha}_{\beta}(t)\sigma^{\beta}_{j}(\widetilde{x}) \tag{2.6b}$$

and therefore

$$\widetilde{N}(t) = N(t) \tag{2.7a}$$

$$\widetilde{N}^{\alpha}(t) = S^{\alpha}_{\beta}(t)(N^{\beta}(t) + P^{\beta}(t)) \tag{2.7b}$$

$$\widetilde{\gamma}_{\alpha\beta}(t) = \Lambda^{\mu}_{\alpha}(t)\Lambda^{\nu}_{\beta}(t)\gamma_{\mu\nu}(t). \tag{2.7c}$$

Thus (2.6) instead of being allocations, turn into a set of highly non-linear partial differential equations. Integrability conditions for this system, i.e. Frobenious' theorem, result in the system (the dot, whenever used, denotes differentiation with respect to time)

$$C^{\beta}_{\mu\nu}\Lambda^{\alpha}_{\beta}(t) = C^{\alpha}_{\kappa\lambda}\Lambda^{\kappa}_{\mu}(t)\Lambda^{\lambda}_{\nu}(t) \tag{2.8a}$$

$$\tfrac{1}{2}\dot{\Lambda}^{\alpha}_{\beta}(t) = C^{\alpha}_{\mu\nu}P^{\mu}(t)\Lambda^{\nu}_{\beta}(t) \tag{2.8b}$$

and 'time-dependent automorphism inducing diffeomorphisms' (AIDs) emerge. The automorphisms of a Lie group G_r form a continuous group. Those members of the group which are continuously connected to the identity element, form a Lie group as well—even though the topology of the latter might be different from that of the former. If one considers parametric families of the automorphic matrices, characterized by the parameters τ^{i}, $\Lambda^{\alpha}_{\beta}(t;\tau^{i})$, and defines

$$\Lambda^{\alpha}_{\beta}(t;\tau^{i})\big|_{\tau^{i}=0} = \delta^{\alpha}_{\beta} \tag{2.9a}$$

$$\left.\frac{\mathrm{d}\Lambda^{\alpha}_{\beta}(t;\tau^{i})}{\mathrm{d}\tau^{i}}\right|_{\tau^{j\neq i}=0} = \lambda^{\alpha}_{\beta(i)} \tag{2.9b}$$

where $\lambda^{\alpha}_{\beta(i)}$ are the generators with respect to the parameter τ^{i} of the Lie algebra of the automorphism group, then from the first of (2.8), after a differentiation with respect to τ^{i}, one gets

$$\lambda^{\alpha}_{\beta(i)}C^{\beta}_{\mu\nu} = \lambda^{\rho}_{\mu(i)}C^{\alpha}_{\rho\nu} + \lambda^{\rho}_{\nu(i)}C^{\alpha}_{\mu\rho}. \tag{2.10}$$

For an extensive treatment on these issues see [27], while for the relation and usage of these generators with conditional symmetries, see [28, 29].

In the $n+1$ decomposition of the spacetime (here $n = 4$), the EFEs in vacuum assume the form

$$E^{0}_{0} = K^{\alpha}_{\beta}K^{\beta}_{\alpha} - K^{2} + R = 0 \tag{2.11a}$$

$$E^{0}_{\alpha} = K^{\mu}_{\nu}C^{\nu}_{\alpha\mu} - K^{\mu}_{\alpha}C^{\nu}_{\mu\nu} = 0 \tag{2.11b}$$

$$E^{\alpha}_{\beta} = \dot{K}^{\alpha}_{\beta} - NKK^{\alpha}_{\beta} + NR^{\alpha}_{\beta} + 2N^{\rho}\left(K^{\alpha}_{\nu}C^{\nu}_{\beta\rho} - K^{\nu}_{\beta}C^{\alpha}_{\nu\rho}\right) = 0 \tag{2.11c}$$

with

$$K^{\alpha}_{\beta}(t) = \gamma^{\alpha\rho}(t) K_{\rho\beta}(t) \tag{2.12a}$$

$$R^{\alpha}_{\beta}(t) = \gamma^{\alpha\rho}(t) R_{\rho\beta}(t) \tag{2.12b}$$

$$K_{\alpha\beta}(t) = -\frac{1}{2N(t)}\left(\dot{\gamma}_{\alpha\beta}(t) + 2\gamma_{\alpha\nu}(t) C^{\nu}_{\beta\rho} N^{\rho}(t) + 2\gamma_{\beta\nu}(t) C^{\nu}_{\alpha\rho} N^{\rho}(t)\right) \tag{2.12c}$$

$$\begin{aligned} R_{\alpha\beta}(t) = {} & C^{\kappa}_{\sigma\tau} C^{\lambda}_{\mu\nu} \gamma_{\alpha\kappa}(t) \gamma_{\beta\lambda}(t) \gamma^{\sigma\nu}(t) \gamma^{\tau\mu}(t) + 2C^{\lambda}_{\alpha\kappa} C^{\kappa}_{\beta\lambda} + 2C^{\mu}_{\alpha\kappa} C^{\nu}_{\beta\lambda} \gamma_{\mu\nu}(t) \gamma^{\kappa\lambda}(t) \\ & + 2C^{\lambda}_{\alpha\kappa} C^{\mu}_{\mu\nu} \gamma_{\beta\lambda}(t) \gamma^{\kappa\nu}(t) + 2C^{\lambda}_{\beta\kappa} C^{\mu}_{\mu\nu} \gamma_{\alpha\lambda}(t) \gamma^{\kappa\nu}(t). \end{aligned} \tag{2.12d}$$

Since GCTs are covariances of the EFEs, the same form of equations (2.11) holds for the transformed quantities,

$$\widetilde{E}^{0}_{0} = E^{0}_{0} = 0 \tag{2.13a}$$

$$\widetilde{E}^{0}_{\alpha} = \Lambda^{\beta}_{\alpha} E^{0}_{\beta} = 0 \tag{2.13b}$$

$$\widetilde{E}^{\alpha}_{\beta} = S^{\alpha}_{\kappa} \Lambda^{\lambda}_{\beta} E^{\kappa}_{\lambda} = 0 \tag{2.13c}$$

where S^{α}_{β} is the inverse of Λ^{α}_{β}. This can be explicitly seen by observing that the extrinsic curvature transforms as a (0, 2) tensor under these transformations, despite the mixing of time and space coordinates. The effect of a time reparametrization is trivially seen also to be a covariance. Finally some terminology is needed: (2.11*a*) is called 'quadratic constraint', (2.11*b*) are called 'linear constraints', and (2.11*c*) are simply the 'equations of motion'.

3. Essential constants

The task of finding the maximal number of essential constants for each model is complicated by the presence of the quadratic and linear constraint equations.

The first thing to observe is that their time derivatives vanish by virtue of the spatial equations of motion; therefore, they are first integrals of motion for these equations and they will be satisfied at all times once they are satisfied at one instant of time. The constraint equations can thus be considered as algebraic relations restricting the initial data at some arbitrarily chosen hypersurface. Accordingly, one initial datum will be absent for each such independent constraint.

The second important thing is that the additive constant of integration at the right-hand side of these constraint equations is identically zero. This points to the fact that the presence of these equations signals the existence of 'gauge' symmetry for the whole system of equations, namely the time-dependent AIDs briefly described in the previous section. Under these transformations, one more constant becomes absorbable. Thus, if we wish to consider the constraints as full fledged (first-class) differential equations, we have to subtract two degrees of freedom (constants in our case) for each such independent equation.

Both points of view are correct and valid: they are nothing but different aspects of the same ingredients of the theory of differential equations. Thus they should yield the same final result concerning the maximal number of essential constants. Below we present the counting algorithms of this number for all 30 (4+1) simply transitive, spatially homogeneous vacuum geometries (see table 1), according to both points of view.

3.1. The initial value theorem

In this section, we apply the initial value theorem to find the maximal number of essential constants each line element should contain in order to describe the entire space of solutions for

Table 1. The structure constants of all four-dimensional, real, Lie algebras.

Lie algebra	Non-vanishing structure constants
$4A_1$	
$A_2 \oplus A_1$	$C^2_{12} = 1$
$2A_2$	$C^2_{12} = 1, C^4_{34} = 1$
$A_{3,1} \oplus A_1$	$C^1_{23} = 1$
$A_{3,2} \oplus A_1$	$C^1_{13} = 1, C^1_{23} = 1, C^2_{23} = 1$
$A_{3,3} \oplus A_1$	$C^1_{13} = 1, C^2_{23} = 1$
$A_{3,4} \oplus A_1$	$C^1_{13} = 1, C^2_{23} = -1$
$A^{\alpha}_{3,5} \oplus A_1, 0 < \lvert\alpha\rvert < 1$	$C^1_{13} = 1, C^2_{23} = \alpha$
$A_{3,6} \oplus A_1$	$C^2_{13} = -1, C^1_{23} = 1$
$A^{\alpha}_{3,7} \oplus A_1, 0 < \alpha$	$C^1_{13} = \alpha, C^2_{13} = -1, C^1_{23} = 1, C^2_{23} = \alpha$
$A_{3,8} \oplus A_1$	$C^1_{23} = 1, C^2_{13} = -1, C^3_{12} = -1$
$A_{3,9} \oplus A_1$	$C^3_{12} = 1, C^1_{23} = 1, C^2_{31} = 1$
$A_{4,1}$	$C^1_{24} = 1, C^2_{34} = 1$
$A^{\alpha}_{4,2}, \alpha \neq (0, 1)$	$C^1_{14} = \alpha, C^2_{24} = 1, C^2_{34} = 1, C^3_{34} = 1$
$A^1_{4,2}$	$C^1_{14} = 1, C^2_{24} = 1, C^2_{34} = 1, C^3_{34} = 1$
$A_{4,3}$	$C^1_{14} = 1, C^2_{34} = 1$
$A_{4,4}$	$C^1_{14} = 1, C^1_{24} = 1, C^2_{24} = 1, C^2_{34} = 1, C^3_{34} = 1$
$A^{\alpha,\beta}_{4,5}, -1 \leqslant \alpha < \beta < 1, \alpha\beta \neq 0$	$C^1_{14} = 1, C^2_{24} = \alpha, C^3_{34} = \beta$
$A^{\alpha,\alpha}_{4,5}, -1 \leqslant \alpha < 1, \alpha \neq 0$	$C^1_{14} = 1, C^2_{24} = \alpha, C^3_{34} = \alpha$
$A^{\alpha,1}_{4,5}, -1 \leqslant \alpha < 1, \alpha \neq 0$	$C^1_{14} = 1, C^2_{24} = \alpha, C^3_{34} = 1$
$A^{1,1}_{4,5}$	$C^1_{14} = 1, C^2_{24} = 1, C^3_{34} = 1$
$A^{\alpha,\beta}_{4,6}, \alpha \neq 0, \beta \geqslant 0$	$C^1_{14} = \alpha, C^2_{24} = \beta, C^3_{24} = -1, C^2_{34} = 1, C^3_{34} = \beta$
$A_{4,7}$	$C^1_{14} = 2, C^2_{24} = 1, C^2_{34} = 1, C^3_{34} = 1, C^1_{23} = 1$
$A_{4,8}$	$C^1_{23} = 1, C^2_{24} = 1, C^3_{34} = -1$
$A^{\beta}_{4,9}, 0 < \lvert\beta\rvert < 1$	$C^1_{23} = 1, C^1_{14} = 1 + \beta, C^2_{24} = 1, C^3_{34} = \beta$
$A^1_{4,9}$	$C^1_{23} = 1, C^1_{14} = 2, C^2_{24} = 1, C^3_{34} = 1$
$A^0_{4,9}$	$C^1_{23} = 1, C^1_{14} = 1, C^2_{24} = 1$
$A_{4,10}$	$C^1_{23} = 1, C^3_{24} = -1, C^2_{34} = 1$
$A^{\alpha}_{4,11}, \alpha > 0$	$C^1_{23} = 1, C^1_{14} = 2\alpha, C^2_{24} = \alpha, C^3_{24} = -1, C^2_{34} = 1, C^3_{34} = \alpha$
$A_{4,12}$	$C^1_{13} = 1, C^2_{23} = 1, C^2_{14} = -1, C^1_{24} = 1$

the given model. In 3+1 dimensions, such a counting has been done some time ago (see, e.g., [4] and the older references therein) using the Behr decomposition of the structure constants $C^{\alpha}_{\beta\gamma}$ for three-dimensional Lie algebras. However, such a decomposition is not known for Lie algebras of dimension 4 or higher. Hence, we have to apply an alternative counting procedure in order to find the essential constants.

A counting which is independent of the dimension can be given using the initial value theorem. This theorem is stated for the (3+1)-dimensional case in, for example, Wald's book [30]. However, it is fairly easily seen that this theorem is valid in any dimension; the arguments in the proof do not depend explicitly on the dimension of the spacetime.

Roughly speaking, the initial value formulation states that a spacetime satisfying the Einstein equations is uniquely determined by specifying the metric, $h_{\alpha\beta}$, and the corresponding extrinsic curvature, $K_{\alpha\beta}$, of an initial spatial hypersurface (i.e. $\gamma_{\alpha\beta}(t_0) = h_{\alpha\beta}$)—at the Gauss normal coordinates system, in which the shift vanishes. The initial data must also satisfy the quadratic constraint and the linear constraint on the initial hypersurface, which are purely algebraic in the initial data. Furthermore, isometric diffeomorphisms on the initial hypersurface can always be extended to isometric diffeomorphisms of the entire spacetime.

The theorem does not mention whether two different initial data can lead to the same spacetime. However, any initial data always generate a 1-parameter family of data which will yield the same maximal development. This 1-parameter family is exactly the time evolution of the pair $(\gamma_{\alpha\beta}(t), K_{\alpha\beta}(t))$. Hence, for a spacetime foliated into spatial hypersurfaces, any hypersurface may serve as an initial hypersurface.

The initial value formulation thus provides us with the following algorithm for counting the essential constants for the spatially homogeneous model of type A:

$$\#(h_{\alpha\beta}, K_{\alpha\beta}) - \dim \text{Aut}(A) - \#(\text{independent constraints}) - 1. \tag{3.1}$$

In our case, $\#(h_{\alpha\beta}, K_{\alpha\beta}) = 20$, since $h_{\alpha\beta}$ and $K_{\alpha\beta}$ are symmetric 2-tensors. Aut(A) is the automorphism group for the Lie algebra A; these automorphisms can be seen as the effect of isomorphic diffeomorphisms on the initial hypersurface. Thus they carry the relevant 'gauge' freedom which must be subtracted [27, 31]. On the initial hypersurface the constraints (quadratic plus linear constraints) are only algebraic equations, thus subtract one for each constraint. Finally, we subtract 1 due to the fact that each initial hypersurface traces out a 1-parameter family of initial data each giving rise to the same spacetime.

Using the above algorithm we produced tables 2 and 3 giving the number of essential constants for all 30, simply transitive, spatially homogeneous vacuum cosmological models of dimension 4+1. Table 2 contains the essential constants for the general form of the algebras, while table 3 contains the essential constants for the exceptional cases in which for some values of the parameters (of the Lie algebra) some of the linear constraints vanish identically.

3.2. Time-dependent AIDs

In this section, we apply the time-dependent AIDs to perform a second independent counting of the maximal number of essential constants each line element should contain in order to describe the entire space of solutions for the given model. This way of counting is valid in any spatial gauge. The key observation is that the solutions to the integrability conditions (2.8) always contain four arbitrary functions of time. These arbitrary functions are distributed in Λ^{α}_{β} and P^{α} in a way that differs for each of the 30 models; for example, to take an extreme case in the Kasner-like model, $4A_1$, Λ^{α}_{β} is completely constant while all four arbitrary functions of time are located in P^{α}. In all cases, P^{α} contains all arbitrary functions through either their derivatives or themselves. Thus two distinct ways of using the gauge freedom suggest themselves, leading to two versions of the counting algorithm.

The first is to use the whole freedom in order to set the shift $\widetilde{N}^{\alpha}$ equal to zero and then see how many first-class linear constraints remain. The corresponding version of the algorithm is

$D = 2 \times (\#\text{ of } \gamma_{\alpha\beta})$
$-2 \times \#\,(\text{linear constraints})$
$-2 \times (\text{the quadratic constraint})$
$-\#\,(\text{parameters of outer automorphic matrices})$
$-(\kappa)$
$\kappa \equiv \dim(\text{Inner}) - \#\text{ functionally independent linear constraints}.$

Peano's theorem requires 2 initial data for each $\gamma_{\alpha\beta}$ since the system is of second order. We subtract 2 constants for each independent first-class constraint. Finally, we subtract the remaining rigid symmetries which are the parameters of the outer automorphisms plus the

Table 2. Essential constants of 4+1 spatially homogeneous models.

Lie algebra	$\#(h_{\alpha\beta}, K_{\alpha\beta})$	# of independent linear constraints	dim Aut(A)	Essential constants
$4A_1$	20	0	16	2
$A_2 \oplus 2A_1$	20	4	8	6
$2A_2$	20	4	4	10
$A_{3,1} \oplus A_1$	20	2	10	6
$A_{3,2} \oplus A_1$	20	4	6	8
$A_{3,3} \oplus A_1$	20	4	8	6
$A_{3,4} \oplus A_1$	20	3	6	9
$A_{3,5}^{\alpha} \oplus A_1, 0 < \lvert\alpha\rvert < 1$	20	4	6	8
$A_{3,6} \oplus A_1$	20	3	6	9
$A_{3,7}^{\alpha} \oplus A_1, 0 < \alpha$	20	4	6	8
$A_{3,8} \oplus A_1 \text{and} A_{3,9} \oplus A_1$	20	3	4	11
$A_{4,1}$	20	3	7	8
$A_{4,2}^{\alpha}, \alpha \neq \{0, 1\}$	20	4	6	8
$A_{4,2}^{1}$	20	4	8	6
$A_{4,3}$	20	4	6	8
$A_{4,4}$	20	4	6	8
$A_{4,5}^{\alpha,\beta}, \alpha, \beta \in [-1, 1) - \{0\}, \alpha \neq \beta$	20	4	6	8
$A_{4,5}^{\alpha,\alpha}, \alpha \in [-1, 1) - \{0\}$	20	4	8	6
$A_{4,5}^{\alpha,1}, \alpha \in [-1, 1) - \{0\}$	20	4	8	6
$A_{4,5}^{1,1}$	20	4	12	2
$A_{4,6}^{\alpha,\beta}, \alpha \neq 0, \beta \geqslant 0$	20	4	6	8
$A_{4,7}$	20	4	5	9
$A_{4,8}$	20	3	5	10
$A_{4,9}^{\beta}, 0 < \lvert\beta\rvert < 1$	20	4	5	9
$A_{4,9}^{1}$	20	4	7	7
$A_{4,9}^{0}$	20	4	5	9
$A_{4,10}$	20	3	5	10
$A_{4,11}^{\alpha}, 0 < \alpha$	20	4	5	9
$A_{4,12}$	20	4	4	10

difference between the number of parameters of the inner automorphisms subgroup and the number of functionally independent linear constraints.

The second consists of all other options, e.g. we can use the functions of time contained in Λ^{α}_{β} to simplify the scale factor matrix $\gamma_{\alpha\beta}(t)$ and the remaining functions contained in P^{α} (if any) to alter somehow the initial shift vector (e.g. equating some components or setting some of them equal to zero). Now the algorithm reads

$D = 2 \times$ (# of $\gamma_{\alpha\beta}$) + $1 \times$ (# of possibly remaining[7] shift vector's components)
$-2 \times$ # (of those linear constraints which do not finally involve shift vector's components)
$-2 \times$ (the quadratic constraint)
$-$# (parameters of those outer automorphic matrices which preserve the form of the reduced $\gamma_{\alpha\beta}$).

For the sake of illustration, we give below three examples of counting with both versions of the algorithm presented in this subsection.

[7] That is, after solving algebraically as many linear constraints as possible—in terms of the shift's vector components—i.e., the shift components which are not expressed in terms of the scale factor matrix components and their derivatives.

Table 3. Essential constants of 4+1 spatially homogeneous exceptional models.

Lie algebra	# $(h_{\alpha\beta}, K_{\alpha\beta})$	# of independent linear constraints	dim Aut(A)	Essential constants
$A^{\alpha}_{3,5} \oplus A_1, 0 < \lvert\alpha\rvert < 1$ for $\alpha = -1/2$	20	3	6	9
$A^{\alpha}_{4,2}$ for $\alpha = -1, -3$	20	3	6	9
$A^{\alpha,\beta}_{4,5}, \alpha, \beta \in [-1, 1) - \{0\}$ for $1 + 2\alpha + \beta = 0$ or $1 + 2\beta + \alpha = 0$	20	3	6	9
$A^{\alpha,\alpha}_{4,5}, \alpha \in [-1, 1) - \{0\}$ for $\alpha = -1$	20	3	8	7
$A^{\alpha,\alpha}_{4,5}, \alpha \in [-1, 1) - \{0\}$ for $\alpha = -1/3$	20	2	8	8
$A^{\alpha,1}_{4,5}, \alpha \in [-1, 1) - \{0\}$ for $\alpha = -1$	20	3	8	7
$A^{\alpha,\beta}_{4,6}, \alpha \neq 0, \beta \geqslant 0$ for $\alpha = -\beta$	20	3	6	9

3.2.1. Type $A_2 \oplus A_1$. The structure constants are $C^2_{12} = 1$. Thus

$$\Lambda^{\alpha}_{\beta}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_5(t) & \lambda_6(t) & 0 & 0 \\ \lambda_9 & 0 & \lambda_{11} & \lambda_{12} \\ \lambda_{13} & 0 & \lambda_{15} & \lambda_{16} \end{pmatrix}$$

$$P^{\alpha}(t) = \left\{ \frac{\lambda_6'(t)}{2\lambda_6(t)}, \frac{-(\lambda_6(t)\lambda_5'(t) - \lambda_5(t)\lambda_6'(t))}{2\lambda_6(t)}, p_3(t), p_4(t) \right\}.$$

Four functions of time appear, as expected; two of them in $\Lambda^{\alpha}_{\beta}(t)$ and correspond to the inner automorphism proper invariant subgroup. The number of functionally independent linear constraints is 4.

First version. We use our entire freedom in order to set the shift vector equal to zero. So:

$\gamma_{\alpha\beta} = 10$
linear constraints in terms of $\dot{\gamma}_{\alpha\beta} = 4$
of parameters of outer automorphic matrices $= 6$
$\kappa = 2 - 4 = -2$
$D = 2 \times 10 - 2 \times 4 - 2 - 6 - (-2) = 6.$

Second version. We use our freedom in order to set $N^3(t) = N^4(t) = 0$ and $\gamma_{12}(t) = 0$, $\gamma_{22}(t) = 1$. So

$\gamma_{\alpha\beta} = 8$
remaining $N^{\alpha} = 0$
linear constraints in terms of $\dot{\gamma}_{\alpha\beta} = 2$
of parameters of those outer automorphic matrices which preserve the form of $\gamma_{\alpha\beta} = 4$
$D = 2 \times 8 - 2 \times 2 - 2 - 4 = 6.$

3.2.2. Type $A_{3,6} \oplus A_1$. The structure constants are $C^2_{13} = -1$ and $C^1_{23} = 1$. Thus

$$\Lambda^\alpha_\beta(t) = \begin{pmatrix} c\cos(f(t)) & c\sin(f(t)) & \lambda_3(t) & 0 \\ -c\sin(f(t)) & c\cos(f(t)) & \lambda_7(t) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda_{15} & \lambda_{16} \end{pmatrix}$$

$$P^\alpha(t) = \left\{ \frac{-(\lambda_3(t)f'(t)) - \lambda_7'(t)}{2}, \frac{-(\lambda_7(t)f'(t)) + \lambda_3'(t)}{2}, \frac{-f'(t)}{2}, p_4(t) \right\}.$$

Four functions of time appear, as expected; the three in $\Lambda^\alpha_\beta(t)$ and correspond to the inner automorphism proper invariant subgroup. The number of functionally independent linear constraints is 3.

First version. We use our entire freedom in order to set the shift vector equal to zero. So

\# $\gamma_{\alpha\beta} = 10$

\# linear constraints in terms of $\dot\gamma_{\alpha\beta} = 3$

\# of parameters of those outer automorphic matrices which preserve the form of $\gamma_{\alpha\beta} = 3$

$\kappa = 3 - 3 = 0$

$D = 2 \times 10 - 2 \times 3 - 2 - 3 = 9.$

Second version. We use our freedom in order to set $N^1(t) = N^2(t) = N^4(t) = 0$ and $\gamma_{12}(t) = 0$. So

\# $\gamma_{\alpha\beta} = 9$

\# remaining $N^\alpha = 0$

\# linear constraints in terms of $\dot\gamma_{\alpha\beta} = 2$

\# of parameters of those outer automorphic matrices which preserve the form of $\gamma_{\alpha\beta} = 3$

$D = 2 \times 9 - 2 \times 2 - 2 - 3 = 9.$

3.2.3. Type $A^{-\frac{1}{3},-\frac{1}{3}}_{4,5}$. The structure constants are $C^1_{14} = 1, C^2_{24} = -\frac{1}{3}$ and $C^3_{34} = -\frac{1}{3}$. Thus

$$\Lambda^\alpha_\beta(t) = \begin{pmatrix} \lambda_1(t) & 0 & 0 & \lambda_4(t) \\ 0 & \frac{c1}{\lambda_1(t)^3} & \frac{c2}{\lambda_1(t)^3} & \lambda_8(t) \\ 0 & \frac{c3}{\lambda_1(t)^3} & \frac{c4}{\lambda_1(t)^3} & \lambda_{12}(t) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^\alpha(t) = \left\{ \frac{-(\lambda_4(t)\lambda_1'(t) - \lambda_1(t)\lambda_4'(t))}{2\lambda_1(t)}, \frac{-(\lambda_8(t)\lambda_1'(t) + 3\lambda_1(t)\lambda_8'(t))}{2\lambda_1(t)}, 8 \leftrightarrow 12, \frac{-\lambda_1'(t)}{2\lambda_1(t)} \right\}.$$

Four functions of time appear, as expected; all the four in $\Lambda^\alpha_\beta(t)$ and correspond to the inner automorphism proper invariant subgroup. The number of functionally independent linear constraints is 2.

First version. We use our entire freedom in order to set the shift vector equal to zero. So

$\gamma_{\alpha\beta} = 10$

linear constraints in terms of $\dot{\gamma}_{\alpha\beta} = 2$

of parameters of those outer automorphic matrices which preserve the form of $\gamma_{\alpha\beta} = 4$

$\kappa = 4 - 2 = 2$

$D = 2 \times 10 - 2 \times 2 - 2 - 4 - 2 = 8.$

Second version. We use our freedom in order to set $\gamma_{11}(t) = 1$ and $\gamma_{14}(t) = \gamma_{24}(t) = \gamma_{34}(t) = 0$. So

$\gamma_{\alpha\beta} = 6$

remaining $N^{\alpha} = 2$

linear constraints in terms of $\dot{\gamma}_{\alpha\beta} = 0$

of parameters of those outer automorphic matrices which preserve the form of $\gamma_{\alpha\beta} = 4$

$D = 2 \times 6 + 2 - 2 - 4 = 8.$

4. Exact solutions

We will here provide examples of spatially homogeneous vacuum solutions in 4+1 dimensions[8]. There are some general things worth noting. For the decomposable cases, $A_3 \oplus A_1$, we can generate vacuum solutions from scalar field solutions of the Bianchi models in 3+1 dimensions. More explicitly, given a vacuum solution in 4+1 dimensions with metric

$$\mathrm{d}s_5^2 = \mathrm{d}s_4^2 + \mathrm{e}^{-2\phi}\,\mathrm{d}\boldsymbol{y}^2 \tag{4.1}$$

the metric $\mathrm{d}\tilde{s}_4^2 = \mathrm{e}^{-\phi}\,\mathrm{d}s_4^2$ will be a solution to the (3+1)-dimensional Einstein equations with a scalar field. Thus, by going the other way, we can construct vacuum solutions in 4+1 dimensions from scalar field solutions in one dimension lower. In many cases (such as type VIII $\oplus\,\mathbb{R}$ and IX $\oplus\,\mathbb{R}$) these are the only non-trivial solutions one knows explicitly (see [1]).

The main object of this section is to give some examples of solutions of the various types. For only two types we know all the possible exact vacuum solutions, the remaining cases we only know some special ones.

4.1. $4A_1 = \mathbf{I} \oplus \mathbb{R}$

There is a 2-parameter family of Kasner solutions which exhaust all solutions of this type [32, 33],

$$\mathrm{d}s^2 = -\mathrm{d}\boldsymbol{t}^2 + \sum_{i=1}^{4} t^{2p_i}\,\mathrm{d}\boldsymbol{x}^i\,\mathrm{d}\boldsymbol{x}^i \tag{4.2}$$

where $\sum_i p_i = \sum_i p_i^2 = 1$.

4.2. $A_2 \oplus 2A_1 = \mathbf{III} \oplus \mathbb{R}$

There is a two-parameter family of plane-wave solutions which can be obtained by restricting the type $\mathrm{VI}_h \oplus \mathbb{R}$ plane waves ($\mathrm{III} = \mathrm{VI}_{-1}$) (see section 4.8).

[8] Some of the solutions are previously known, even though in many cases the true number of free parameters was not recognized.

Also, there is a 2-parameter family of solutions, with a higher symmetry, given by

$$\mathrm{d}s^2 = -\frac{k^2\omega^2\exp(-2(1+a)t)\,\mathrm{d}\boldsymbol{t}^2}{\sinh^4\omega t} + \frac{k^2\exp(-2(1+a)t)}{\sinh^2\omega t}(\mathrm{d}\boldsymbol{x}^2 + \mathrm{e}^{-2x}\,\mathrm{d}\boldsymbol{y}^2) + \mathrm{e}^{2at}\,\mathrm{d}\boldsymbol{z}^2 + \mathrm{e}^{2t}\,\mathrm{d}\boldsymbol{w}^2 \tag{4.3}$$

$$\omega^2 = a^2 + a + 1.$$

The symmetry group of these solutions is $G_5 = SL(2,\mathbb{R}) \times \mathbb{R}^2$ which acts transitively on the spatial hypersurfaces.

4.3. $2A_2$

There is one solution which can be obtained by a Wick rotation of a solution in [35]:

$$\mathrm{d}s^2 = -\mathrm{d}\boldsymbol{t}^2 + \frac{t^2}{3}[(\mathrm{d}\boldsymbol{x}^2 + \mathrm{e}^{2x}\,\mathrm{d}\boldsymbol{y}^2) + (\mathrm{d}\boldsymbol{z}^2 + \mathrm{e}^{2z}\,\mathrm{d}\boldsymbol{w}^2)]. \tag{4.4}$$

This has indeed the bigger symmetry group $G_6 = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acting on the spatial hypersurfaces Σ_t. It is also algebraically special of type 22 in the sense of [35].

4.4. $A_{3,1} \oplus A_1 = \mathbf{II} \oplus \mathbb{R}$

The general solutions (containing six parameters) are not known to our knowledge, but we have found a 4-parameter family of solutions[9]. It is given by

$$\mathrm{d}s^2 = -\frac{a_4}{\omega}\exp((a_1 + a_2 + 3a_3)t)\cosh\omega t\,\mathrm{d}\boldsymbol{t}^2 + \frac{\omega}{a_4}\frac{\mathrm{e}^{-a_3 t}}{\cosh\omega t}(\mathrm{d}\boldsymbol{x} - z\,\mathrm{d}\boldsymbol{y})^2 + \mathrm{e}^{a_3 t}\frac{\cosh\omega t}{\omega}(\mathrm{e}^{a_1 t}\,\mathrm{d}\boldsymbol{y}^2 + \mathrm{e}^{a_2 t}\,\mathrm{d}\boldsymbol{z}^2) + \mathrm{e}^{2a_3 t}\,\mathrm{d}\boldsymbol{w}^2 \tag{4.5}$$

where $\omega^2 = a_1a_2 + 2(a_1 + a_2)a_3 + a_3^2$.

This family of solutions generalizes Taub's type II vacuum solutions.

4.5. $A_{3,2} \oplus A_1 = \mathbf{IV} \oplus \mathbb{R}$

A 3-parameter family of plane-wave solutions is given by equation (4.7) with $s = 2\beta_+$.

4.6. $A_{3,3} \oplus A_1 = \mathbf{V} \oplus \mathbb{R}$

A 2-parameter family of plane-wave solutions is given by equation (4.15) with $s = 2\beta_+$.

There is also a 3-parameter family of solutions given by

$$\mathrm{d}s^2 = -\frac{k^2\omega^2\,\mathrm{e}^{-a_1 t}\,\mathrm{d}\boldsymbol{t}^2}{4\sinh^3\omega t} + \mathrm{e}^{2a_1 t}\,\mathrm{d}\boldsymbol{w}^2 + \frac{k^2\,\mathrm{e}^{-a_1 t}}{\sinh\omega t}(\mathrm{e}^{a_2 t}\,\mathrm{e}^{-2z}\,\mathrm{d}\boldsymbol{x}^2 + \mathrm{e}^{-a_2 t}\mathrm{e}^{-2z}\,\mathrm{d}\boldsymbol{y}^2 + \mathrm{d}\boldsymbol{z}^2) \tag{4.6}$$

$$3\omega^2 = 3a_1^2 + a_2^2.$$

4.7. $A_{3,4} \oplus A_1 = \mathbf{VI_0} \oplus \mathbb{R}$

There is a 1-parameter family of solutions given by equation (4.19) with $p = -1$ and $q = 0$.

[9] See also [34] which considers the $A_{3,1} \oplus A_1$ and $A_{3,3} \oplus A_1$ cases.

4.8. $A^p_{3,5} \oplus A_1 = \mathbf{VI}_h \oplus \mathbb{R}$

A 3-parameter family of plane-wave solutions is given by equation (4.15) with $s = 2\beta_+$. There is also a 2-parameter family of solutions given by equation (4.18) with $q = 0$.

4.9. $A_{3,6} \oplus A_1 = \mathbf{VII}_0 \oplus \mathbb{R}$

Apart from the solutions with higher symmetry deducible from scalar field Bianchi type VII_0, the authors do not know of any other non-trivial solutions.

4.10. $A^p_{3,7} \oplus A_1 = \mathbf{VII}_h \oplus \mathbb{R}$

A 3-parameter family of plane-wave solutions is given in equation (4.26) with $s = 2\beta_+$.

4.11. $A_{3,8} \oplus A_1 = \mathbf{VIII} \oplus \mathbb{R}$

Apart from the solutions with higher symmetry deducible from scalar field Bianchi type VIII, the authors do not know of any other non-trivial solutions (see also [36]).

4.12. $A_{3,9} \oplus A_1 = \mathbf{IX} \oplus \mathbb{R}$

Apart from the solutions with higher symmetry deducible from scalar field Bianchi type IX, the authors do not know of any other non-trivial solutions (see also [36]).

4.13. $A_{4,1}$

No vacuum solutions of this type are known to the authors[10].

4.14. $A^p_{4,2}$

There is a 3-parameter family of plane-wave solutions for $p > -2$ [38],

$$\begin{aligned}
\mathrm{d}s^2 = {} & \mathrm{e}^{2t}(-\mathrm{d}\boldsymbol{t}^2 + \mathrm{d}\boldsymbol{w}^2) + \exp(2s(w+t)) \\
& \times \Bigg[\exp(-4\beta_+(w+t)) \bigg(\mathrm{d}\boldsymbol{x} + \frac{Q_1}{P_1} \exp(3\beta_+(w+t))\, \mathrm{d}\boldsymbol{y} \\
& + [A + B(w+t)] \exp(3\beta_+(w+t))\, \mathrm{d}\boldsymbol{z} \bigg)^2 \\
& + \exp(2\beta_+(w+t))\, (\mathrm{d}\boldsymbol{y} + Q_3(w+t)\, \mathrm{d}\boldsymbol{z})^2 + \exp(2\beta_+(w+t))\, \mathrm{d}\boldsymbol{z}^2 \Bigg]
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
& s(1-s) = 2\beta_+^2 + \tfrac{1}{6}\left(Q_1^2 + Q_2^2 + Q_3^2\right) \\
& P_1 = 3\beta_+
\end{aligned} \tag{4.8}$$

and

$$A = \frac{3\beta_+ Q_2 - Q_1 Q_3}{3\beta_+} \qquad B = \frac{Q_1 Q_3}{3\beta_+}. \tag{4.9}$$

[10] There are some known self-similar solutions with a perfect fluid [6]. Note that there is a typo in equation (83); all exponents should be divided by γ.

The group parameter is given by

$$p = \frac{s - 2\beta_+}{s + \beta_+}. \tag{4.10}$$

For $p = -2$, there is a 1-parameter family of vacuum solutions due to Demaret and Hanquin[11] [37]:

$$\begin{aligned} ds^2 = {} & k^2 e^{3t^2} t^{-\frac{1}{24}} \left(t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)(-d\boldsymbol{t}^2 + d\boldsymbol{w}^2) + t^{\frac{2}{3}} e^{4w} d\boldsymbol{x}^2 \\ & + t^{\frac{5}{3}} \left(t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right) e^{-2w} d\boldsymbol{y}^2 + \frac{t^{-\frac{1}{3}}}{t^{-\frac{1}{2}} + t^{\frac{1}{2}}} e^{-2w} (d\boldsymbol{z} - w\, d\boldsymbol{y})^2. \end{aligned} \tag{4.11}$$

4.15. $A^1_{4,2}$

There is a 2-parameter family of plane-wave solutions if one sets $Q_1 = 0$, and then $\beta_+ = 0$ in equation (4.7).

4.16. $A_{4,3}$

Plane-wave solutions for the Lie algebra type $A_{4,3}$ can be obtained by taking the $p \longrightarrow \infty$ limit of $A^p_{4,2}$. In this limit we get $\beta_+ = -s$ and thus the metric can be written as [38]

$$\begin{aligned} ds^2 = {} & e^{2t}(-d\boldsymbol{t}^2 + d\boldsymbol{w}^2) + \exp(6s(w+t)) \\ & \times \left(d\boldsymbol{x} + \frac{Q_1}{P_1} \exp(-3s(w+t))\, d\boldsymbol{y} + [A + B(w+t)] \exp(-3s(w+t))\, d\boldsymbol{z}\right)^2 \\ & + (d\boldsymbol{y} + Q_3(w+t)\, d\boldsymbol{z})^2 + d\boldsymbol{z}^2 \end{aligned} \tag{4.12}$$

where

$$s = \frac{1}{6}\left(1 \pm \sqrt{1 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)}\right) \tag{4.13}$$

and A, B are given in equation (4.9) with $\beta_+ = -s$.

4.17. $A_{4,4}$

There is a 3-parameter set of plane-wave solutions given by [38]

$$\begin{aligned} ds^2 = {} & e^{2t}(-d\boldsymbol{t}^2 + d\boldsymbol{w}^2) + \exp(2s(w+t)) \\ & \times \left[\left(d\boldsymbol{x} + Q_1(w+t)\, d\boldsymbol{y} + (w+t)\left[Q_2 + \frac{Q_1 Q_3}{2}(w+t)\right] d\boldsymbol{z}\right)^2 \right. \\ & \left. + (d\boldsymbol{y} + Q_3(w+t)\, d\boldsymbol{z})^2 + d\boldsymbol{z}^2 \right] \end{aligned} \tag{4.14}$$

where

$$s(1-s) = \tfrac{1}{6}\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

[11] However, they did not realize that the solution was part of a 1-parameter family of solutions.

4.18. $A_{4,5}^{pq}$

Given $p+q+1>0$, a 3-parameter set of plane-wave solutions can be given by [38]

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{e}^{2t}(-\mathrm{d}\boldsymbol{t}^2+\mathrm{d}\boldsymbol{w}^2) + \exp(2s(w+t))\Bigg[& \exp(-4\beta_+(w+t)) \\ & \times \left(\mathrm{d}\boldsymbol{x} + \frac{Q_1}{P_1}\exp(P_1(w+t))\,\mathrm{d}\boldsymbol{y} + \frac{Q_1Q_3+P_3Q_2}{P_3P_2}\exp(P_2(w+t))\,\mathrm{d}\boldsymbol{z}\right)^2 \\ & + \exp(2(\beta_+ + \sqrt{3}\beta_-)(w+t))\left(\mathrm{d}\boldsymbol{y} + \frac{Q_3}{P_3}\exp(P_3(w+t))\,\mathrm{d}\boldsymbol{z}\right)^2 \\ & + \exp(2(\beta_+ - \sqrt{3}\beta_-)(w+t))\,\mathrm{d}\boldsymbol{z}^2\Bigg] \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} & s(1-s) = 2\left(\beta_+^2+\beta_-^2\right) + \tfrac{1}{6}\left(Q_1^2+Q_2^2+Q_3^2\right) \\ & P_1 = 3\beta_+ + \sqrt{3}\beta_- \qquad P_2 = 3\beta_+ - \sqrt{3}\beta_- \qquad P_3 = -2\sqrt{3}\beta_-. \end{aligned} \tag{4.16}$$

The group parameters are related to these parameters as follows:

$$p = \frac{s+(\beta_+ + \sqrt{3}\beta_-)}{s+(\beta_+ - \sqrt{3}\beta_-)} \qquad q = \frac{s-2\beta_+}{s+(\beta_+ - \sqrt{3}\beta_-)}. \tag{4.17}$$

There are also some other solutions due to Demaret and Hanquin [37][12]. Given $p+q+1 \neq 0$, then there is a 2-parameter family of vacuum solutions,

$$\begin{aligned} \mathrm{d}s^2 = {} & k^2(\sinh t)^{2\sum P_i^2}\left(\tanh\frac{t}{2}\right)^{2\sum\alpha_i P_i}(-\mathrm{d}\boldsymbol{t}^2+\mathrm{d}\boldsymbol{w}^2) \\ & + \sum(\sinh t)^{2P_i}\left(\tanh\frac{t}{2}\right)^{2\alpha_i}\mathrm{e}^{2P_i w}(\mathrm{d}\boldsymbol{x}^i)^2 \end{aligned} \tag{4.18}$$

where $\sum P_i = 1$, $\sum \alpha_i = 0$ and $\sum \alpha_i^2 = 1 + \sum P_i^2$.

Given $p+q+1=0$, there is a 1-parameter family of solutions due to Demaret and Hanquin [37]:

$$\mathrm{d}s^2 = k^2\exp\left((1+p^2+q^2)\frac{t^2}{2}\right)t^{-\frac{2}{3}}(-\mathrm{d}\boldsymbol{t}^2+\mathrm{d}\boldsymbol{w}^2) + t^{\frac{2}{3}}(\mathrm{e}^{2w}\,\mathrm{d}\boldsymbol{x}^2 + \mathrm{e}^{2pw}\,\mathrm{d}\boldsymbol{y}^2 + \mathrm{e}^{2qw}\,\mathrm{d}\boldsymbol{z}^2). \tag{4.19}$$

Also, for the exceptional case $A_{4,5}^{pq*}$ $(q=-(1+p)/2)$, we have found a self-similar solution which generalizes the Collinson–French type $\mathrm{VI}^*_{-1/9}$ vacuum,

$$\begin{aligned} \mathrm{d}s^2 = {} & -\mathrm{d}\boldsymbol{t}^2 + t^2\,\mathrm{d}\boldsymbol{x}^2 + \left[t^{\frac{(1-p)^2}{b}}\exp(-\sqrt{6}(1+p)rx)\,\mathrm{d}\boldsymbol{y} + \frac{1}{2r\sqrt{b}}t\,\mathrm{d}\boldsymbol{x}\right]^2 \\ & + t^{\frac{6(1+p)}{b}}\exp(4\sqrt{6}rx)\,\mathrm{d}\boldsymbol{z}^2 + t^{\frac{6p(1+p)}{b}}\exp(4p\sqrt{6}rx)\,\mathrm{d}\boldsymbol{w}^2 \end{aligned} \tag{4.20}$$

where

$$r = \frac{\sqrt{1+p+p^2}}{5p^2+2p+5} \qquad b = 5p^2+2p+5. \tag{4.21}$$

[12] They only give it as a 1-parameter family.

4.19. $A_{4,5}^{p,p}$

This is a special case of the above. For the plane-wave solutions, equation (4.15), one has to set $Q_1 = 0$, and then $P_1 = 0$.

4.20. $A_{4,5}^{p,1}$

Similarly as in the above case, but now set $Q_2 = 0$ and then $P_2 = 0$ in equation (4.15).

In addition to this, we have found a 3-parameter family of solutions for the particular value $p = -1$. It is given by

$$\begin{aligned} \mathrm{d}s^2 = {} & -\frac{\omega^2 k^2 \exp(-(4a_1+2a_2)t)\,\mathrm{d}\boldsymbol{t}^2}{\sinh^8 \omega t} + \frac{\mathrm{e}^{-a_1 t}\,\mathrm{e}^{-2w}}{\sinh^2 \omega t}(\mathrm{e}^{-a_2 t}\,\mathrm{d}\boldsymbol{x}^2 + \mathrm{d}\boldsymbol{z}^2) \\ & + \exp((2a_1+a_2)t)\,\mathrm{e}^{2w}\sinh^2\omega t\,\mathrm{d}\boldsymbol{y}^2 + \frac{k^2\exp(-(4a_1+2a_2)t)\,\mathrm{d}\boldsymbol{w}^2}{\sinh^6 \omega t} \qquad (4.22) \\ 8\omega^2 = {} & 3a_1^2 + 3a_1 a_2 + a_2^2. \end{aligned}$$

4.21. $A_{4,5}^{1,1}$

The whole set of solutions is in this case known[13]. The set is two dimensional and the general solution is given by equation (4.18) with the restriction $P_1 = P_2 = P_3 = 1/3$. Explicitly,

$$\begin{aligned} \mathrm{d}s^2 = {} & k^2 \sinh^{\frac{2}{3}} t(-\mathrm{d}\boldsymbol{t}^2 + \mathrm{d}\boldsymbol{w}^2) \\ & + \sinh^{\frac{2}{3}} t\,\mathrm{e}^{\frac{2}{3}w}\left[\left(\tanh\frac{t}{2}\right)^{2a_1}\mathrm{d}\boldsymbol{x}^2 + \left(\tanh\frac{t}{2}\right)^{2a_2}\mathrm{d}\boldsymbol{y}^2 + \left(\tanh\frac{t}{2}\right)^{-2(a_1+a_2)}\mathrm{d}\boldsymbol{z}^2\right] \\ 2 = {} & 3a_1^2 + 3a_2^2 + 3a_1 a_2. \end{aligned} \qquad (4.23)$$

4.22. $A_{4,6}^{pq}$

Again we have plane-wave solutions [38]. Let s be given by

$$s(1-s) = 2\beta_+^2 + \tfrac{2}{3}\omega^2\sinh^2 2\beta + \tfrac{1}{6}\left(Q_1^2 + Q_2^2\right). \qquad (4.24)$$

Define also the two one-forms:

$$\begin{aligned} \boldsymbol{\omega}^2 &= \cos[\omega(w+t)]\,\mathrm{d}\boldsymbol{y} - \sin[\omega(w+t)]\,\mathrm{d}\boldsymbol{z} \\ \boldsymbol{\omega}^3 &= \sin[\omega(w+t)]\,\mathrm{d}\boldsymbol{y} + \cos[\omega(w+t)]\,\mathrm{d}\boldsymbol{z}. \end{aligned} \qquad (4.25)$$

The plane-wave solutions of type $A_{4,6}^{pq}$ can now be written as

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{e}^{2t}(-\mathrm{d}\boldsymbol{t}^2 + \mathrm{d}\boldsymbol{w}^2) + \exp(2s(w+t)) \\ & \times\left[\exp(-4\beta_+(w+t))\{\mathrm{d}\boldsymbol{x} + \exp(3\beta_+(w+t))(q_1\mathrm{e}^{-\beta}\boldsymbol{\omega}^3 - q_2\,\mathrm{e}^{\beta}\boldsymbol{\omega}^2)\}^2\right. \\ & \left. + \exp(2\beta_+(w+t))\left\{\mathrm{e}^{-2\beta}(\boldsymbol{\omega}^2)^2 + \mathrm{e}^{2\beta}(\boldsymbol{\omega}^3)^2\right\}\right] \end{aligned} \qquad (4.26)$$

where

$$q_1 = \frac{Q_1\omega + 3\beta_+ Q_2\,\mathrm{e}^{2\beta}}{\omega^2 + 9\beta_+^2} \qquad q_2 = \frac{Q_2\omega - 3\beta_+ Q_1\,\mathrm{e}^{-2\beta}}{\omega^2 + 9\beta_+^2}. \qquad (4.27)$$

The group parameters are related to these constants via

$$p = \frac{\beta_+(s-2\beta_+)}{\omega(s+\beta_+)} \qquad q = \frac{\beta_+}{\omega}. \qquad (4.28)$$

[13] All solutions with a γ-law non-tilted perfect fluid are also known, see [6].

4.23. $A_{4,7}$

No such solutions are known to the authors.

4.24. $A_{4,8}$

There is a solution which is the $p \to -1$ limit of the metric (4.29).

4.25. $A_{4,9}^{p}$

There is a simple power-law solution for each $-1 < p \leqslant 1$,

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}\boldsymbol{t}^2 + t^2\,\mathrm{d}\boldsymbol{w}^2 + k^2 t^{\frac{2(2p^2+5p+2)}{3(p^2+p+1)}} \exp(-2(p+1)\sigma w)(\mathrm{d}\boldsymbol{x} - z\,\mathrm{d}\boldsymbol{y})^2 \\ + t^{\frac{2(p+2)^2}{3(p^2+p+1)}}\,\mathrm{e}^{-2\sigma w}\,\mathrm{d}\boldsymbol{y}^2 + t^{\frac{2(2p+1)^2}{3(p^2+p+1)}}\,\mathrm{e}^{-2p\sigma w}\,\mathrm{d}\boldsymbol{z}^2 \end{aligned} \tag{4.29}$$

where

$$\sigma^2 = \frac{7p^2 + 13p + 7}{6(p^2 + p + 1)^2} \qquad k^2 = \frac{2(7p^2 + 13p + 7)}{9(p^2 + p + 1)}. \tag{4.30}$$

Due to the power-law dependence, this solution is self-similar.

There is one special case worth noting, namely $p = -1/2$.[14] In this case the metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}\boldsymbol{t}^2 + \tfrac{3}{2}t^2(\mathrm{d}\boldsymbol{w}^2 + \mathrm{e}^{-2w}\,\mathrm{d}\boldsymbol{y}^2) + \mathrm{e}^{-w}(\mathrm{d}\boldsymbol{x} - z\,\mathrm{d}\boldsymbol{y})^2 + \mathrm{e}^{w}\,\mathrm{d}\boldsymbol{z}^2. \tag{4.31}$$

Note that the spatial hypersurfaces are fibre bundles over $\mathbb{H}^2$. In fact, the symmetry group is larger for this metric than one would expect; it is the semi-direct product $G_5 = \mathbb{R}^2 \ltimes SL(2, \mathbb{R})$ with a $U(1)$ stabilizer.

4.26. $A_{4,9}^{1}$

There is a solution obtained from equation (4.29) by setting $p = 1$, which is fairly interesting [39]. By a rescaling of the coordinates the solution can be written as

$$\mathrm{d}s^2 = -\mathrm{d}\boldsymbol{t}^2 + \frac{t^2}{2}\left[\mathrm{d}\boldsymbol{w}^2 + \mathrm{e}^{-2w}\left(\mathrm{d}\boldsymbol{x} + \frac{1}{2}(y\,\mathrm{d}\boldsymbol{z} - z\,\mathrm{d}\boldsymbol{y})\right)^2 + \mathrm{e}^{-w}(\mathrm{d}\boldsymbol{y}^2 + \mathrm{d}\boldsymbol{z}^2)\right]. \tag{4.32}$$

In this case the spatial surfaces are isometric to the complex hyperbolic space, $\mathbb{H}^2_{\mathbb{C}}$, and hence, it has an eight-dimensional isometry group, $G_8 = PU(2, 1)$, acting multiply transitive on the spatial surfaces (it has a $U(2)$ stabilizer).

4.27. $A_{4,9}^{0}$

There is a solution obtained from equation (4.29) by setting $p = 0$.

4.28. $A_{4,10}$

This algebra acts simply transitive on $\mathsf{Nil}^3 \times \mathbb{R}$ [39], so all solutions of the type II $\oplus\, \mathbb{R}$ admitting an extra symmetry acting on the spatial surfaces, are also invariant under this algebra. Hence, the solutions (4.5) with $a_1 = a_2$ are invariant under this group. Solutions with $A_{4,10}$ as a maximal symmetry are not known to the authors.

[14] This corresponds to a Lie algebra acting simply transitive on the model geometry $\mathbb{F}^4$ [39].

4.29. $A^{p}_{4,11}$

The solution (4.32) is invariant under this group due to the fact that this algebra acts simply transitive on $\mathbb{H}^2_{\mathbb{C}}$ [39]. Other solutions are not known to the authors.

4.30. $A_{4,12}$

This algebra acts simply transitive on $\mathbb{H}^3 \times \mathbb{R}$ [39] so all solutions having this higher symmetry group are invariant under $A_{4,12}$. An interesting example—although far from general—is the 1-parameter family of solutions obtained by Wick-rotating the (5D) five-dimensional Schwarzschild solution:

$$\mathrm{d}s^2 = -\frac{t^2\,\mathrm{d}t^2}{t^2+2M} + \frac{1}{t^2}(t^2+2M)\,\mathrm{d}x^2 + t^2[\mathrm{d}y^2 + \mathrm{e}^{-2y}(\mathrm{d}z^2 + \mathrm{d}w^2)]. \qquad (4.33)$$

As is clearly seen, this solution has a far larger symmetry group than $A_{4,12}$, namely $G_7 = SL(2,\mathbb{C}) \times \mathbb{R}$. However, solutions with a maximal symmetry group $A_{4,12}$ are not known to the authors.

5. Conclusion

We have shown that the usage of the automorphism group is a very efficient way of identifying the true gravitational degrees of freedom for a simply transitive spatially homogeneous vacuum geometry. Many investigations have suffered from the failure of identifying these. In particular, if we wish to find the *general* solution under a given set of assumptions, then it is essential to *ab initio* identify the number of true degrees of freedom. At this point, we deem as appropriate to state that the time-dependent AIDs were not only used to derive the second counting algorithm, but also to find some of the solutions exhibited in section 4.

In this paper, we specifically used this method to find the dimension of the set of all Ricci-flat spatially homogeneous models of dimension 4+1. Our main results are given in tables 2 and 3.

Inspecting tables 2 and 3 it is seen that the most general types have 11 essential constants. Hence, in order to specify a certain solution under the above assumptions, we need to specify up to 11 parameters. The maximal number of parameters happens for the following two types:

$$A_{3,8} \oplus A_1 = \mathrm{VIII} \oplus \mathbb{R} \qquad A_{3,9} \oplus A_1 = \mathrm{IX} \oplus \mathbb{R}.$$

Interestingly, these two algebras are the trivial extensions of the Bianchi-type Lie algebras VIII and IX and not some indecomposable ones—as one might have expected. It is also noteworthy that, the set of the allowed numbers of the essential constants does not contain the numbers 1, 3, 4 and 5. This does not occur in 3+1 dimensions where the various models saturate all the range of values between 1 and 4. There, the models with the minimum number of essential constants are the Kasner (type I) and Joseph (type V). The corresponding 4+1 counterpart of type I, i.e. $4A_1$ algebra is seen—by means of the algorithm—to contain 2 essential constants. Thus why the number 1 is excluded. In fact this 'hole' increases with the dimension, since the corresponding Abelian types, will have $d-2$ essential constants, in d+1 dimensions. On the other hand, the 4+1 counterparts of the next 'minimal' 3+1 models (type V and its 'neighbour' type II with 1 and 2 essential constants respectively) i.e. the algebras $A_{3,3} \oplus A_1$ and $A_{3,1} \oplus A_1$ have both 6 essential constants. The reason for this is that the number of the 'would be constants' depends not only on the more components of the scale factor matrix $\gamma_{\alpha\beta}(t)$ but also on the number of the linear constraints (the last being depended on the algebra). Thus from 2 the number of essential constants is lifted up to 6. Thus why the numbers between them, i.e.

3, 4, 5 are also excluded. This sort of 'irregularity' does not occur for the rest of the cases, and thus all the numbers from 6 to 11 appear.

We have also given some exact solutions, some of which are believed to be new. Only in two cases ($4A_1$ and $A_{4,5}^{1,1}$) the posited line element is the most general one. For the remaining types only special solutions are known. However, some of them—such as the self-similar ones—may serve as asymptotes for more general solutions (this does, however, require a stability analysis within the class under consideration which to date is only done for the plane-wave solutions [20]).

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